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Canonical Transformations, Umbral Calculus, and Orthogonal Theory

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We present a theory of representations of canonical transformations which links together the theories of evolution semigroups, Hamiltonian flows, umbral operators, and cross-sequences. We also present a different approach to the orthogonal theory of moment systems.

1. INTRODUCTION AND PRELIMINARIES

In a series of papers, [2–5], and mainly in [3], Feinsilver has presented an operator calculus of evolution semigroups and classical Hamiltonian mechanics; i.e., a way of computing the solution of

$$\frac{\partial u}{\partial t} = H(\nabla_x)\, u(x, t), \qquad x \in \mathbb{R}^n,\ t > 0 \tag{1.1}$$

knowing the solution of the Hamilton pair

$$\dot{x} = \nabla_p H, \qquad \dot{p} = -\nabla_x H \tag{1.2}$$

where $H = H(p)$, and $H(p)$ is assumed to be analytic in p. It is shown in [3], for example, that

$$u(x, t) = f(C^+(t))\, 1(x)$$

where $C^+(t) = x + tv(-\nabla_x)$, $v(p) = \nabla_p H$. Our notation differs slightly from Feinsilver's. One of the areas to which the formalism applies is to the theory of moment systems (polynomials of binomial type, Scheffer sequences, cross-sequences). An application of the canonical formalism—the theory of canonical transformations—to these problems comes up in [3–5] in operational form.

In [6] a theory of representations of canonical transformations was introduced to study how equations of the type (1.1) for $H = H(x, p)$ behave under canonical transformations. Here we develop some of it in the setup

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of [3, 4], first to reobtain known results and thus test the different approach, and second, to have a technique allowing us to treat these problems when $H = H(x, p)$.

In Section 2 we study an integral representation of a special class of canonical transformations, which happens to include the umbral transforms of [8, 10]. This provides a missing link between [3, 4] and [8–10]. In Section 3 we anayze the moment systems problem and provide another approach the orthogonal theory of [3, 4].

Let us present some basics about canonical transformations. A rapid exposé can be found in [9] and an in-depth study in [1]. We shall say that the (generating) function $F(x, p, t)$, defined and smooth on $\mathbb{R}^{2n+1}$, generates the canonical transformation $(x, p) \to (Q, P)$ if the equations

$$p = \nabla_x F, \qquad Q = \nabla_p F \tag{1.3}$$

can be solved for (Q, P) in terms of (x, p) and vice versa. The system (1.2) in the new coordinates (Q, P) becomes

$$\dot{Q} = \nabla_P \tilde{H}, \qquad \dot{P} = -\nabla_Q \tilde{H} \tag{1.4}$$

where

$$\tilde{H} = H + \partial F/\partial t. \tag{1.5}$$

LEMMA 1.6. *Let $F_1(x_1, p_2)$ and $F_2(x_2, p_3)$ generate the canonical transformations $(x_1, p_1) \to (x_2, p_2)$ and $(x_2, p_2) \to (x_3, p_3)$. Then $F(x_1, p_3) = F_1(x_1, p_2) - x_2 p_2 + F(x_2, p_3)$ generates the composite transformation $(x_1, p_1) \to (x_3, p_3)$.*

Proof. Easy.

Comment. One uses (1.3) to eliminate p_2 and x_2!

LEMMA 1.7. *Let $F(x_1, p_2, t) = x_1 \cdot U(p_2) + \phi(p_2, t)$ generate $(x_1, p_1) \to (x_2, p_2)$, where $U: \mathbb{R}^n \to \mathbb{R}^n$ is smoothly invertible, and ϕ smooth. Then $F'(x_2, p_1, t) = x_2 \cdot V(p_1) - \phi(V(p_1), t)$ generates the inverse transformation, $V = U^{-1}$.*

Proof. Use 1.6 and verify that the composition of the two transformations is the generator of the identity transformation, $x_1 \cdot p_2$ or $x_2 \cdot p_1$ depending on the order of application.

We shall say a few words about polynomial sequences. For the theory the reader should go to [10] or to [8] for the multivariable case. There is [11], of course, but the language of [8] or [10] is more related to ours. We shall use the conventional multiindex notation throughout; i.e., given

$(k_1,\ldots, k_n)$ in N^k, $(a_1,\ldots, a_n)$ in $\mathbb{R}^n$. Then $a^k = a_1^{k_1} a_2^{k_2} \cdots a_n^k$; $(\nabla_x)^k = \partial^{k_1}/\partial x_1^{k_1} \cdots \partial^{k_2}/\partial x_n^k$. Also $k \leqslant m$ will mean $k_1 \leqslant m_i$, $i = 1,\ldots, n$,

$$\binom{m}{k} = \binom{m_1}{k_1} \cdots \binom{m_n}{k_n}.$$

We denote by e_i the standard ith basis vector and as usual $k! = k_1! \cdots k_n!$.

It is shown in [8] or [10] that there is a correspondence between polynomials $\{u_k(x)\}_{k \in N^n}$ of binomial type and functions U: $\mathbb{R}^k \to \mathbb{R}^u$, invertible with respect to composition. It is given by

$$e^{x \cdot U(\xi)} = \sum_k u_k(x)\, \xi^k/k!. \tag{1.8}$$

Also, if $V = U^{-1}$, then $V_i(\nabla_x)\, u_k(x) = k_i u_{(k-e_i)}(x)$.

Scheffer sequences $\{J_k\}$, or shift basis for $V(\nabla_x)$, correspond to generating functions of the type

$$\sum_k J_k \xi^k/k! = \exp\{x \cdot U(\xi) + f(\xi)\} \tag{1.9}$$

with $f(\xi)$ some analytic function of ξ in $\mathbb{R}^n$. And cross-sequences are polynomial sequences $\{J_k(x, t)\}$ with generating function

$$\sum J_k(x, t)\, \xi^k/k! = \exp\{x \cdot U(\xi) + tf(\xi)\}. \tag{1.10}$$

2. Transformation Theory

In this section we are going to analyze a bit the integral representations of the canonical transformations generated by functions of the type $F(x, P) = x \cdot U(P)$ with invertible U: $\mathbb{R}^u \to \mathbb{R}^n$. More general cases with different applications were treated in [6].

The generating functions $F(x, P) = x \cdot U(P)$, $F(Q, p) = Q \cdot V(p)$ generate the transformations

$$\begin{aligned} (x, p) &\to (Q, P) = (xJ, V(p)) \\ (Q, P) &\to (x, p) = (QJ^{-1}, U(P)) \end{aligned} \tag{2.1}$$

where

$$J_{ij}(p) = \frac{\partial U_i}{\partial P_j}(V(p)), \qquad J_{ij}^{-1}(P) = \frac{\partial V_i}{\partial p_j}(U(P)).$$

Let $f(x)$: $\mathbb{R}^n \to \mathbb{R}^n$ and *define* its transform $\tilde{f}(Q)$ by

$$f(Q) = \int e^{-Q \cdot V(ik)} f(k)\, dk/(2\pi)^n. \tag{2.2}$$

LEMMA 2.3. *With the notations above*

$$f(x) = \int e^{-x \cdot U(ik)} \hat{\tilde{f}}(k)\, dk/(2\pi)^n.$$

Proof.

$$\int \frac{e^{-x \cdot U(ik)}}{(2\pi)^n}\, dk \int e^{ik \cdot Q}\, dQ \int \frac{e^{-Q \cdot V(ik')}}{(2\pi)^n}\, \hat{f}(k')\, dk$$

$$= \int dk' \int e^{-x \cdot U(ik)}\, \delta(x - iV(ik'))\, dk\, \hat{f}(k')/(2\pi)^n$$

$$= \int dk'\, e^{-ik' \cdot x} \hat{f}(k')/(2\pi)^n = f(x).$$

Comment. The lemma asserts the invertibility of the transformation on the class of "functions" where it is defined.

Associated to (2.1) we have the induced transformation on the "measures" that integrate the class of functions on which (2.1) is defined.

If $\int f(x)\, \mu(dx)$ is defined, *define* $\tilde{\mu}(dQ)$ by

$$\int \tilde{\mu}(dQ)\, \tilde{f}(Q) = \int f(x)\, \mu(x) \tag{2.4}$$

and again we are skipping existence questions.

Note that

$$\int f(x)\, \mu(x) = \int \frac{\hat{f}(k)\, dk}{(2\pi)^n} \int e^{-ik \cdot x} \mu(dx)$$

$$= \int \hat{f}(k)\, \hat{\mu}(-k)\, dk/(2\pi)^n$$

and that, using (2.2),

$$\int \tilde{f}(Q)\, \tilde{\mu}(dQ) = \int \tilde{\mu}(iV(ik))\, \hat{f}(k)\, dk/(2\pi)^n$$

and therefore, from (2.4) we should have $\hat{\mu}(k) = \hat{\tilde{\mu}}(iV(-ik))$, or

$$\hat{\tilde{\mu}}(k) = \hat{\mu}(-iU(-ik)). \tag{2.5}$$

LEMMA 2.6. *Let $\mu_t(dx)$ be a convolution semigroup of measures on $\mathbb{R}^n$. Define $\tilde{\mu}_t(dQ)$ by* (2.4). *If $\hat{\mu}_t(k)=\exp t\bar{H}(ik)$ then $\hat{\tilde{\mu}}=\exp t\bar{\tilde{H}}(ik)$ with $H(k)=\tilde{H}(U(k))$.*

Proof. Just apply (2.5).

What this says is that $\tilde{P}_t\tilde{f}(Q)=\int \tilde{f}(Q+Q')\,\tilde{\mu}_t(dQ)$ has generator $\tilde{H}(-\nabla_Q)$ if $P_t f(x)=\int f(x+x')\,\mu(dx')$ has generator $H(-\nabla_x)$, or in other words, $\tilde{u}(Q,t)=\int \{\exp -Q\cdot V(ik)\}\,\hat{u}(k,t)\,dk/(2\pi)^n$ satisfies $\partial\tilde{u}/\partial t=\tilde{H}(-\nabla_Q)\,\tilde{u}$ if $u(x,t)=P_t f$ satisfies $\partial u/\partial t=H(-\nabla_x)\,u$.

An extension of Lemma 2.6 is

LEMMA 2.7. *With the definitions above,*

$$\int \mu(dx)\,P_t f(x)=\int \tilde{\mu}(dQ)\,\tilde{P}_t\tilde{f}(Q).$$

Proof. It suffices to consider $f(x)=\exp ik\cdot x$. In this case it is clear that $P_t f(x)=\exp ik\cdot x+t\bar{H}(ik)$ and $\tilde{f}(Q)=\exp -Q\cdot V(ik)$.

Therefore

$$\tilde{P}_t\tilde{f}(Q)=\exp Q\cdot V(ik)+t\bar{H}(ik)$$

since $\bar{\tilde{H}}(V(ik))=\bar{H}(U(V(ik)))=\bar{H}(ik)$ and thus

$$\int \mu(dx)\,e^{ik\cdot x+t\bar{H}(ik)}=\hat{\mu}(k)\,e^{t\bar{H}(ik)}$$

$$\int \tilde{\mu}(dQ)\,e^{-Q\cdot\bar{V}(ik)+t\bar{H}(ik)}=\hat{\tilde{\mu}}(i\bar{V}(ik))\,e^{t\bar{H}(ik)}$$

and the result follows from (2.5).

Let us now see how to transform polynomials of binomial type and cross-sequences via transforming their generating functions. To begin with, note that the transform of $\exp i\alpha\cdot Q=\tilde{f}(Q)$, the generation function of $\{Q^n\}$, is

$$f(x)=\int e^{-x\cdot U(ik)}\,\hat{\tilde{f}}(k)\,dk/(2\pi)^n=\int e^{-x\cdot U(ik)}\,\delta(k-\alpha)\,dk$$

$$=\exp x\cdot(-\bar{U}(i\alpha))=\sum_n u_n(x)(i\alpha)^n/n!.$$

We thus see that the transforms introduced in Section 2 contain the umbral operators of [8] or [10].

Instead of beginning with $\{Q^m\}$, we could have started with any sequence $\{\tilde{P}_m(Q)\}$ of binomial type, with generating function $\exp Q \cdot R(i\alpha)$. When canonically transformed the sequence becomes $\{P_m(x)\}$ generated by $\exp -x \cdot U(R(i\alpha))$.

A simple extension of these facts is the following. If $\{\tilde{P}_m(Q, t)\}$ is a cross-sequence with generating function. $\exp iQ \cdot \alpha + t\tilde{M}(-i\alpha) = \sum (i\alpha)^m \tilde{P}_m(Q, t)/m!$, the transformed polynomial sequence and its generating function are

$$\exp x \cdot \bar{U}(i\alpha) + t\tilde{M}(i\alpha) = \exp x \cdot \bar{U}(i\alpha) + t\bar{M}(U(i\alpha))$$

$$= \sum (i\alpha)^m \tilde{P}_m(x, t)/m!$$

where $M = \tilde{M} \cdot V$.

Besides the correspondence of umbral transforms with canonical transforms, we have the correspondence of cross-sequences, see (1.10), with evolution semigroups via $P_t f(x) = \int f(x+y)\,\mu_t(dy) = \exp ik \cdot x + t\bar{H}(ik)$ for $f(x) = \exp ik \cdot x$. The correspondence with completely integrable Hamiltonian systems is the subject of [2–4] and is touched upon in the next section.

We also have the following expansion.

LEMMA 2.8. *Let $\tilde{f}(Q)$ and $f(x)$ be related as in* (2.2). *Let $\{V_n(Q)\}$ be the sequence generated by* $\exp Q \cdot (-\bar{V}(ik))$, *then*

$$\tilde{f}(Q) = \sum v_m(Q)\, f^{(m)}(0)/m!,$$

Proof.

$$\tilde{f}(Q) = \int e^{-Q \cdot V(ik)} \hat{f}(k)\, dk/(2\pi)^n$$

$$= \sum v_m(Q)(m!)^{-1} \int (-ik)^m \hat{f}(k)\, dk/(2\pi)^n$$

$$= \sum v_m(Q)\, f^{(m)}(0)/m!.$$

Comment. Analogously, $f(x) = \sum U_m(x)\, \tilde{f}^{(m)}(0)/m!$.

Comment. This throws some light on the meaning of the $f \to \tilde{f}$ transform: in the Taylor expansion of f about $x = 0$, replace x^n by $v_n(Q)$, which is its image under the transformation.

In the next section we shall explore the "dynamical content" of these transformations. To close this section let us examine how the multiplication by Q and the gradient ∇_Q transform under the correspondence $\tilde{f} \to f$ given in (2.4).

To begin with, notice that

$$\begin{aligned} V_l(-\nabla_x) f(x) &= \int V_l(U(ik))\, e^{-x \cdot U(ik)} \hat{\tilde{f}}(k)\, dk/(2\pi)^n \\ &= \int e^{-x \cdot U(ik)} (-ik_l)\, \hat{\tilde{f}}(k)\, dk/(2\pi)^n \\ &= \int e^{-x \cdot U(ik)} \left(-\frac{\partial}{\partial Q_e} \tilde{f}\right)^{\wedge} (k)\, dk/(2\pi)^n. \end{aligned} \tag{2.9}$$

Also, an application of the chain rule shows that

$$\int e^{-x \cdot U(ik)} (Q_l f)^{\wedge} (k)\, dk/(2\pi)^n = x_j \frac{\partial U_j}{\partial P_l} (-\nabla_x) f(x) \tag{2.10}$$

which asserts that in order to obtain the "operator" associated to $Q_l = Q_l(x, p) = x_j\, \partial U_j/\partial P_l$ replace p by $-\nabla_x$.

Since the transformation (x, p) into (Q, P) is canonical, i.e., the Poisson brackets $[Q_i, P_j] = \delta_{ij}$ when computed in terms of the (x, p) variables, it is satisfying to verify that

$$\left[x_j \frac{\partial U_j}{\partial P_l}(\nabla_x),\ V_m(\nabla_x)\right] f(x) = \delta_{lm} f(x). \tag{2.11}$$

This asserts that the correspondence

$$\begin{aligned} &Q \to \text{multiplication by } Q \\ &P \to -\nabla_Q \end{aligned}$$

transforms covariantly with respect to the representation of the canonical transformation $(Q, P) \to (x, p)$ by means of $\tilde{f} \to f$ as in (2.4). This fact is implicitly made use of in [5], for example.

3. Orthogonal Theory

Consider a Hamiltonian system, with Hamiltonian $H(p)$. The solution to the pair (1.2), through (x, p), is

$$\begin{aligned} x(t) &= x + tv(p), \qquad v(p) = \nabla_p H(p) \\ p(t) &= p. \end{aligned}$$

In the (Q, P) coordinates, the new Hamiltonian is, see (1.4),

$$\tilde{H}(P) = H(U(P))$$

and in the new coordinates, the trajectory through (Q, P) is

$$Q(t) = Q + t\tilde{v}(P), \qquad \tilde{v} = \nabla_p \tilde{H}(P)$$
$$P(t) = P.$$

Certainly $Q(t) = Jx(t)$, with J defined at the beginning of Section 2. Proceeding as Feinsilver, define

$$\begin{aligned} C^+(t) &= x + tv(-\nabla_x), \qquad & C(t) &= \nabla_x \\ \tilde{C}^+(t) &= Q + t\tilde{v}(-\nabla_Q), \qquad & \tilde{C}(t) &= \nabla_Q \end{aligned} \tag{3.1}$$

and introduce, for $(m) = (m_1, ..., m_n)$,

$$\begin{aligned} h_{(n)}(x, t) &= (C(t))^{(m)}\, \Omega_0(x), \\ \tilde{h}_{(m)}(Q, t) &= (\tilde{C}(t))^m\, \tilde{\Omega}_0(x) \end{aligned} \tag{3.2}$$

where $\Omega_0(x) = 1$ and $\tilde{\Omega}_0(Q) = e^{Q \cdot V(0)} = 1$ if $U(0) = V(0) = 0$ which we assume from now on.

The following proposition is taken from [3, 4].

PROPOSITION 3.3. *With the notations introduced above*

(a) $[C_i, C_j^+] = \delta_{ij}$, $\qquad [\tilde{C}_i, \tilde{C}_j^+] = \delta_{ij}$, etc.

(b) $C_i^+ h_m(x, t) = h_{(m+e_i)}(x, t)$, $\qquad \tilde{C}_i^+ \tilde{h}_m(Q, t) = \tilde{h}_{m+e_i}(Q, t)$

(c) $C_i h_m(x, t) = m_i h_{(m-e_i)}(x, t)$, $\qquad \tilde{C}_i \tilde{h}_m(Q, t) = m_i \tilde{h}_{(m-e_i)}(Q, t)$

(d) $C_i^+ C_i h_m(x, t) = m_i h_m(x, t)$, $\qquad \tilde{C}_i^+ \tilde{C}_i \tilde{h}_m(Q, t) = m_i \tilde{h}_m(Q, t)$

(e) $\dfrac{\partial h_m}{\partial t} = H(-\nabla_x)\, h_m$, $\qquad \dfrac{\partial \tilde{h}_m}{\partial t} = \tilde{H}(-\nabla_Q)\, \tilde{h}_{(m)}(Q, t)$.

Another version of the comments preceding (2.8) is contained in

PROPOSITION 3.4.

$$J_m(x, t) = \int e^{x \cdot U(ik)} \hat{\tilde{h}}_m(k, t)\, dk/(2\pi)^n \tag{3.5}$$

then

(a) $J_m(x, t) = \int \delta(x - y)\, J_m(y, dy) = \int \tilde{\mu}_x(dQ)\, \tilde{h}_m(Q, t)$,

(b) $\sum_m (-\bar{V}(ik))^m J_m(x, t)/m! = \exp\{ik \cdot x + t\bar{H}(ik)\}$.

Proof. (a) This is just a simple application of Lemma 2.7. To prove (b) proceed as

$$\sum_m \frac{(-V(ik))^m}{m!} J_m(x,t) = \int \frac{e^{x\cdot U(ik')}}{(2\pi)^n} e^{t\bar{H}(ik')}$$

$$\times \int e^{ik'Q} e^{-Q\cdot V(ik)}\, dQ\, dk'$$

$$= \int e^{x\cdot U(ik') + t\bar{H}(ik')} \delta(k' + i\bar{V}(ik))\, dk'$$

$$= \exp\{ik\cdot x + t\bar{H}(ik)\}.$$

Part (b) asserts that $J_m(x,t)$ can be obtained by expanding $\exp ik\cdot x + t\bar{H}(ik)$ in powers of $-\bar{V}(ik)$. This is the point of departure from Feinsilver, and, for example, in [4] by using generating function techniques, he proposes criteria or conditions on H and $U = V^{-1}$ for $J_n(x,-t)$ to be an orthogonal family with respect to the density of the semigroup P_t.

Before going on, let us write a representation for the $J_n(x,t)$ analogous to that obtained in the last section of Chapter 4 of [3]. To begin with, let $m(t)$ and $\tilde{m}(t)$ be the moments of a pair of measures $\mu_t(dx)$ and $\tilde{\mu}_t(dQ)$ related as in Lemma 2.6. As in [3]

$$h_m(x,t) = \sum_{l\le k} \binom{k}{l} m_{k-l}(t)\, x^l$$

$$\tilde{h}_m(Q,t) = \sum_{l\le k} \binom{k}{l} \tilde{m}_{k-l}(t)\, Q^l.$$

Then from the comment right after Lemma 2.8 and Proposition 3.4 it follows that

$$J_m(x,t) = \sum_{l\le k} \binom{k}{l} U_l(x)\, \tilde{m}_{k-l}(t). \tag{3.6}$$

Let us now obtain the J_m as linear combinations of the h_m, and translate that representation in terms of the "process" X_t associated to the semigroup $\exp tH$.

PROPOSITION 3.7. *$J_m(x,t) = \sum_{k\le n} C_k^m h_k(x,t)$, with*

$$C_k^m = (\nabla_a)^m (U(a))^m|_{a=0}/k! \qquad \textit{for} \quad k \le m. \tag{3.8}$$

Proof.

$$J_m(x,t) = \int \exp x \cdot (-U(ik))\, \hat{\hat{h}}_m(k,t)\, dk/(2\pi)^n$$

$$= \int \exp x \cdot (-U(ik))$$

$$\times \int \exp ik \cdot Q(\exp t\tilde{H}(-\nabla_a)\, Q^m 1)\, dQ\, dk/(2\pi)^n$$

$$= \int \exp x \cdot (-U(ik)) \exp t\tilde{H}(ik) \int e^{iQ \cdot k} Q^m\, dQ\, dk/(2\pi)$$

$$= \int \exp x \cdot (-U(ik)) \exp t\tilde{H}(ik)(\nabla_{ik})^m\, \delta(k)\, dk$$

$$= (\nabla_{ik})^m\, e^{x \cdot (-U(ik))} e^{t\tilde{H}(ik)}|_{k=0}$$

$$= (\nabla_a)^m\, e^{-x \cdot U(e) + tH(U(a))}|_{a=0}$$

$$= (\nabla_a)^m \sum_{k \geqslant 0} \frac{(U(a))^k}{k!} h_k(x,t)|_{a=0}.$$

Since U is invertible $\det|\partial U/\partial a| \neq 0$ which implies that for $k > m$ $(\nabla_a)^m\, (U(a))^k|_{a=0} = 0$ and therefore the conclusion follows.

Comments. (i) $J_m(x,t)$ are both polynomials of degree $|m|$.

(ii) Since $U_i(a) = \sum_{m \geqslant 1} C^m_{e_i} a^m$ we conclude that the change of basis 3.7 determines $U(a)$ uniquely, $m > 1$ meaning $|m| \geqslant 1!$.

Since $h_m(x,t) = P_t x^m(x) = EX^m_t = \int (x+y)^m\, \mu_t(dy)$, then

$$J_m(x,t) = \sum_{k \leqslant m} C^m_k E^x X^k_t = E^x \sum_{k \leqslant m} C^m_k X^k_t.$$

This suggests we define

$$Y_m(t) = \sum_{k \leqslant m} C^m_k X^k_t \tag{3.9}$$

which can be restated as

PROPOSITION 3.10. *For each multiindex m*

$$Y_m(t) = U_m(X_t) \tag{3.11}$$

where the $U_m(x)$ are the polynomials of binomial type generated by $\exp x \cdot U(a)$.

Proof. Note that

$$Y_{(m)}(t) = (\nabla_a)^m \sum_k (U(a))^k X_t^k/k!|_{a=0} = (\nabla_a)^m e^{X_t \cdot U(a)}|_{a=0}$$

$$= (\nabla_a)^m \sum_{k \geq 0} a^k U_k(X_t)/k!|_{a=0} U_m(X_t).$$

For some more mathematical precision as to the meaning of the process X_t when $\exp tH$ is not a positive semigroup, the reader can check with [7]. Observe that

$$\int J_m J_{m'} P_t(dx) = E \; Y_m(t) \, Y_{m'}(t) \tag{3.12}$$

which suggests

PROPOSITION 3.13. *Consider the process X_t and let $\{C_k^m\}$, m, k in $\mathbb{N}^n$, be a sequence such that* (a) $Y_m(t) = \sum_{k \leq m} C_k^m X_t^k$ *and* (b) $E \; Y_m(t) \, Y_{m'}(t) = \delta_{m,m'} E \; Y_m^2(t)$. *Then, if* $C_0^0 = 1$ *and* $\det\{C_{e_j}^{e_i}\} \neq 0$, *there is an invertible mapping* $U: \mathbb{R}^n \to \mathbb{R}^n$, $U(0) = 0$, *such that* $J_m(x, t) = E^x Y_m(t)$ *and* $h_m(x, t) = E^x X_t^m$ *are related by* 3.7. *Also the* $J_m(x, t)$ *can be obtained from* $h_m(Q, t) = \tilde{E}^Q X_t^m$ *as in* (3.5). *Here X_t is the "process" associated to* $\exp t\tilde{H}(\nabla_Q)$ *and* $H = H \cdot U$.

Proof. Scattered in what we have said above. We shall only add that $\det\{C_{e_j}^{e_i}\} \neq 0$ implies invertibility of $U = (U_1, \ldots, U_n)$ with $U_i(a) = \sum_{m \geq 1} C_{e_i}^m a$.

Comments. (i) It is clear that orthogonalizing the powers $\{X_t^n\}$ with respect to the measure dP^x on sample space is the same as orthogonalizing the set $\{x^n\}$ with respect to the distribution $P_t(dx)$ of X_t on $\mathbb{R}^n$.

(ii) One case in which Proposition 3.13 does not work corresponds to $H(p) = -\frac{1}{3}p^3$, p in $\mathbb{R}$. This seems to be due to the nonpositivity of the transition kernel corresponding to $\exp tH(-\nabla)$. But the proposition applies to all processes with independent increments, and these cases do not seem to be fully taken care of by the procedure described in [3, 4]. This raises two questions. First, if it is true that only "processes" corresponding to convolution semigroup of positive measures can be orthogonalized, how come Feinsilver does not obtain them? Second, and this is the other side of the coin, do there exist convolution semigroups of nonpositive measures having orthogonal systems associated to them?

To close, we mention the simplest example in which the procedure works. Take $H(p) = p^2/2$. Orthogonalizing the set $\{x^n\}$ with respect to $(2\pi t)^{-1/2} \exp -x^2/2t$ yields the usual hermite polynomials

$$H_n(x, t) = \sum \binom{n}{2k} (-t)^k x^{n-2k} (2k)!/2^k k!.$$

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